Solusi UTS Matematika B (19 Maret 2016)

Bagian A

1. $\int \frac{3x+2}{x^2(x+1)}dx$.

$$\begin{aligned}\frac{3x+2}{x^2(x+1)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1} = \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)} \\ &= \frac{(A+C)x^2+(A+B)x+B}{x^2(x+1)} \\ 3x+2 &= (A+C)x^2+(A+B)x+B\end{aligned}$$

   Maka

$$A+C=0, \quad A+B=3, \quad B=2$$

   Maka diperoleh $B=2, A=1, C=-1$.

$$\int \frac{3x+2}{x^2(x+1)}dx = \int\left(\frac{1}{x}-\frac{1}{x+1}+\frac{2}{x^2}\right)dx = \ln|x|-\ln|x+1|-\frac{2}{x}+C.$$

2. $\lim_{x\to\frac{\pi}{4}} \frac{x-\frac{\pi}{4}}{1-\tan x} \underset{0/0}{\overset{l'H}{=}} \lim_{x\to\frac{\pi}{4}} \frac{1}{-\sec^2 x} = \lim_{x\to\frac{\pi}{4}} -\cos^2 x = -\left(\frac{\sqrt{2}}{2}\right)^2 = -\frac{1}{2}$

3. $\int_0^2 \frac{dx}{\sqrt{2-x}}$ merupakan integral tak wajar. Jadi, $\int_0^2 \frac{dx}{\sqrt{2-x}} = \lim_{a\to 2^-}\int_0^a \frac{dx}{\sqrt{2-x}}$. Misalkan $u(x)=2-x$. Maka

$$\int \frac{dx}{\sqrt{2-x}} = -\int u^{-\frac{1}{2}}du = -2\sqrt{u} = -2\sqrt{2-x}+C$$

   Maka.

$$\lim_{a\to 2^-}\int_0^a \frac{dx}{\sqrt{2-x}} = \lim_{a\to 2^-}\left[-2\sqrt{2-x}\right]_0^a = \lim_{a\to 2^-} -2\sqrt{2-a}+2\sqrt{2-0} = 2\sqrt{2}$$

4. Misalkan $f(x)=e^{3x}$. Maka $f'(x)=3e^{3x}, f''(x)=9e^{3x}$, dan $f'''(x)=27e^{3x}$ dan

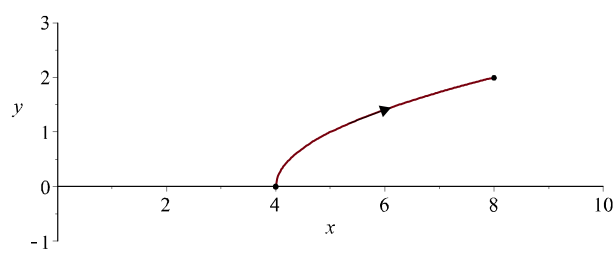
$$f'(0)=3, \quad f''(0)=9, \quad \text{dan} \quad f'''(x)=27$$

   Jadi, polinom Maclaurin bagi e^{3x} adalah

$$P_3(x) = 1+\frac{3}{1!}x+\frac{9}{2!}x^2+\frac{27}{3!}x^3 = 1+3x+\frac{9}{2}x^2+9x^3.$$

5. $x(t)=3+t$ dan $y(t)=\sqrt{t-1}$. Maka $y^2=t-1=(x-3)-1$ atau $x=y^2+4$. Persamaan kartesius adalah

$$x=y^2+4, \quad \text{dan } y\geq 0.$$



6. $\vec{r}(t) = \left\langle \frac{1}{\sqrt{16-t^2}}, \ln(t-1)\right\rangle$, terdefinisi jika $16-t^2>0$ dan $t-1>0$ yaitu

$$-4<t<4 \text{ dan } t>1$$

   Maka daerah definisi adalah $(1,4)=\{t: 1<t<4\}$.

7. $A=(2,-1,2)$, $B=(-2,3,2)$. Misalkan $C=(0,y,0)$. Maka $\overrightarrow{CA}=(2,-1-y,2)$, $\overrightarrow{CB}=(-2,3-y,2)$, $\overrightarrow{AB}=(-4,4,0)$. Segitiga ABC siku-siku di C jika

$$0=\overrightarrow{CA}\cdot\overrightarrow{CB}=-4+(-1-y)(3-y)+4=(y+1)(y-3)$$

yaitu jika $y=1$ dan $y=3$. Jadi, $C=(0,1,0)$ atau $C=(0,3,0)$
Segitiga ABC siku-siku di B jika

$$0=\overrightarrow{CB}\cdot\overrightarrow{AB}=(-2,3-y,2)\cdot(-4,4,0)=20-4y$$

yaitu jika $y=5$. Jadi, $C=(0,5,0)$
Segitiga ABC siku-siku di A jika

$$0=\overrightarrow{CA}\cdot\overrightarrow{AB}=(2,-1-y,2)\cdot(-4,4,0)=-4y-12,$$

yaitu jika $y=-3$. Jadi, $C=(0,-3,0)$

Bagian B

1. (a) $\lim_{t\to\infty}\left(10-te^{-t}\right)=10-\lim_{t\to\infty}\frac{t}{e^t}\underset{\infty/\infty}{\overset{l'H}{=}}10-\lim_{t\to\infty}\frac{1}{e^t}=10.$

   (b)

$$C=\int_0^\infty (k-f(t))\,dt=\int_0^\infty\left(10-\left(10-te^{-t}\right)\right)dt=\int_0^\infty te^{-t}dt=\lim_{b\to\infty}\int_0^b te^{-t}dt$$

$$\int te^{-t}dt\underset{dv=e^{-t}dt,v=-e^{-t}}{\overset{u=t}{=}}\int udv=-te^{-t}-\int -e^{-t}dt=-te^{-t}+\int e^{-t}dt=-te^{-t}-e^{-t}+C=-\frac{t+1}{e^t}+C$$

   Maka

$$C=\lim_{b\to\infty}\int_0^b te^{-t}dt=\lim_{b\to\infty}\left(-\frac{b+1}{e^b}\right)--\frac{0+1}{e^0}=1-\lim_{b\to\infty}\frac{b+1}{e^b}\underset{\infty/\infty}{\overset{l'H}{=}}1-0=1$$

2. $f(x)=\ln(2x-1)$. Maka

$$\begin{array}{ll}
f'(x)=\frac{2}{2x-1} & f'(1)=2\\
f''(x)=-\frac{2\times 2}{(2x-1)^2} & f''(1)=-4\\
f'''(x)=\frac{2^3(2!)}{(2x-1)^3} & f'''(1)=2^3 2!\\
f^{(4)}(x)=-\frac{2^4 3!}{(2x-1)^4} & f^{(4)}(1)=-2^4 3!
\end{array}$$

Maka secara umum $f^{(n)}(x)=\frac{(-1)^{n+1}2^n(n-1)!}{(2x-1)^n}$. Polinom Taylor sekitar $a=1$ adalah

$$2(x-1)-4(x-1)^2+\frac{8}{3}(x-1)^3-4(x-1)^4+\cdots$$

Maka $\ln(1.1)=\ln(2(1.05)-1)=f(1.05)$.

$$R_n(1.05)=\left|\frac{(-1)^n 2^{n+1}}{(2c-1)^{n+1}}\frac{n!}{(n+1)!}(1.05-1)^{n+1}\right|=\left|\frac{2^{n+1}}{(2c-1)^{n+1}}\frac{1}{n+1}\right||1.05-1|^{n+1}$$

dengan c antara 1 dan 1.05 sehingga $1<2c-1<1.1$. Jadi,

$$R_n(1.05)<\frac{2^{n+1}}{n+1}\frac{10^{-(n+1)}}{2^{n+1}}=\frac{10^{-(n+1)}}{n+1}$$

Maka $R_n(1.05)<5\times10^{-3}$ jika $\frac{10^{-(n+1)}}{n+1}<5\times10^{-3}=\frac{10^{-2}}{2}$. Maka pilih $n=1$. Jadi, nilai hampiran adalah

$$\ln(1.1)\approx 2((1.05)-1)=0.1$$

3. Diketahui $\vec{a}\,(t) = \left\langle \frac{1}{2}, e^{-t}, \sin t \right\rangle$ dengan $\vec{v}\,(0) = \langle 3, 1, 1 \rangle$ dan $\vec{r}\,(0) = \langle -1, 0, 3 \rangle$

(a) $\vec{v}\,(t) = \left\langle \int \frac{1}{2} dt, \int e^{-t} dt, \int \sin t dt \right\rangle = \left\langle \frac{t}{2} + C_1, -e^{-t} + C_2, -\cos t + C_3 \right\rangle.$

$$\vec{v}\,(0) = \langle C_1, -1 + C_2, -1 + C_3 \rangle = \langle 3, 1, 1 \rangle.$$

Maka $C_1 = 3, C_2 = 2, C_3 = 2$.

$$\vec{v}\,(t) = \left\langle \frac{t}{2} + 3, -e^{-t} + 2, -\cos t + 2 \right\rangle$$

Selanjutnya $\vec{r}\,(t) = \left\langle \int \left(\frac{t}{2} + 3\right) dt, \int \left(-e^{-t} + 2\right) dt, \int \left(-\cos t + 2\right) dt \right\rangle = \left\langle \frac{t^2}{4} + 3t + D_1, 2t + e^{-t} + D_2, 2t - \sin t + D \right.$

$$\vec{r}\,(0) = \langle D_1, 1 + D_2, D_3 \rangle = \langle -1, 0, 3 \rangle.$$

Maka $D_1 = -1, D_2 = -1$, dan $D_3 = 3$.

$$\vec{r}\,(t) = \left\langle \frac{t^2}{4} + 3t - 1, e^{-t} + 2t - 1, -\sin t + 2t + 3 \right\rangle.$$

(b) Misalkan $\theta =$ sudut antara $\vec{r}\,(0)$ dan $\vec{a}\,(0)$. $\vec{a}\,(0) = \left\langle \frac{1}{2}, 1, 0 \right\rangle.$

$$\cos\theta = \frac{\langle -1, 0, 3 \rangle \cdot \left\langle \frac{1}{2}, 1, 0 \right\rangle.}{\|\langle -1, 0, 3 \rangle\| \left\| \left\langle \frac{1}{2}, 1, 0 \right\rangle. \right\|} = \frac{-\frac{1}{2}}{\sqrt{1+9}\sqrt{\frac{1}{4}+1}} = -\frac{1}{\sqrt{50}} = -\frac{1}{5\sqrt{2}}$$